

# SEPARATORS OF ARITHMETICALLY COHEN-MACAULAY FAT POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

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**ABSTRACT.** Let  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  be a set of fat points that is also arithmetically Cohen-Macaulay (ACM). We describe how to compute the degree of a separator of a fat point of multiplicity  $m$  for each point in the support of  $Z$  using only a numerical description of  $Z$ . Our formula extends the case of reduced points which was previously known.

## 1. INTRODUCTION

Fix an algebraically closed field  $k$  of characteristic zero. Let  $R = k[x_0, x_1, y_0, y_1]$  be the  $\mathbb{N}^2$ -graded polynomial ring with  $\deg x_i = (1, 0)$  for  $i = 0, 1$  and  $\deg y_i = (0, 1)$  for  $i = 0, 1$ . The ring  $R$  is the coordinate ring of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Consider now a set of points  $X = \{P_1, \dots, P_s\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ , and fix positive integers  $m_1, \dots, m_s$ . The goal of this note is to study some of the properties of the scheme  $Z = m_1P_1 + \dots + m_sP_s$  of fat points (precise definitions are deferred until the next section). In particular, we are interested in describing the separator of  $P_i$  of multiplicity  $m_i$ .

Recall that for sets of points  $X = \{P_1, \dots, P_s\} \subseteq \mathbb{P}^n$ , a homogeneous form  $F \in k[\mathbb{P}^n]$  is called a **separator** of  $P \in X$  if  $F(P) \neq 0$ , but  $F(Q) = 0$  for all  $Q \in X \setminus \{P\}$ . Over the years, a number of authors have shown how to exploit information about the separator of a point to describe properties of the set of reduced points  $X \subseteq \mathbb{P}^n$  (e.g., see [1, 2, 3, 4, 11, 13, 14]). In a series of papers, the authors, along with Marino, (see [6, 10]) generalized some of these results by studying separators of fat points, a family of non-reduced points. Roughly speaking, a **separator of a point  $P_i$  of multiplicity  $m_i$**  and the **degree of a point  $P_i$  of multiplicity  $m_i$**  are defined in terms of the generators of  $I_{Z'}/I_Z$  in  $R/I_Z$  where  $I_{Z'}$  is the defining ideal of  $Z' = m_1P_1 + \dots + (m_i - 1)P_i + \dots + m_sP_s$ .

General properties of separators of both reduced points and fat points in a multiprojective space were studied in [8, 9, 10, 12]. We now specialize to the case of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . By restricting to this case, we can improve upon the results found in [10]. The main result of this paper (Theorem 3.4) is to show how to compute the degree of a point  $P_i$  of multiplicity  $m_i$  directly from the combinatorics of the scheme, i.e., the number of points on the various rulings of  $\mathbb{P}^1 \times \mathbb{P}^1$  and the multiplicities, provided that the scheme is ACM. This result generalizes the reduced case as found in [12, Theorem 7.4] and [9, Theorem 4.4].

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## 2. PRELIMINARIES AND NOTATION

**2.1. ACM fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ .** Let  $X$  be a set of  $s$  distinct points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  denote the projection morphism defined by  $P \times Q \mapsto P$ . Let  $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$  be the other projection morphism. The set  $\pi_1(X) = \{P_1, \dots, P_a\}$  is the set of  $a \leq s$  distinct first coordinates that appear in  $X$ . Similarly,  $\pi_2(X) = \{Q_1, \dots, Q_b\}$  is the set of  $b \leq s$  distinct second coordinates. For  $i = 1, \dots, a$ , let  $L_{P_i}$  be the degree  $(1, 0)$  form that vanishes at all the points with first coordinate  $P_i$ . Similarly, for  $j = 1, \dots, b$ , let  $L_{Q_j}$  denote the degree  $(0, 1)$  form that vanishes at points with second coordinate  $Q_j$ .

Let  $D := \{(x, y) \mid 1 \leq x \leq a, 1 \leq y \leq b\}$ . If  $P \in X$ , then  $I_P = (L_{R_i}, L_{Q_j})$  for some  $(i, j) \in D$ . So, we can write each point  $P \in X$  as  $P_i \times Q_j$  for some  $(i, j) \in D$ . (Note: this does not mean that if  $(i, j) \in D$ , then  $P_i \times Q_j \in X$ ; there may exist a tuple  $(i, j) \in D$ , but  $P_i \times Q_j \notin X$ .)

Suppose that  $X$  is a set of distinct points in  $\mathbb{P}^1 \times \mathbb{P}^1$  and  $|\pi_1(X)| = a$  and  $|\pi_2(X)| = b$ . Let  $I_{P_i \times Q_j} = (L_{R_i}, L_{Q_j})$  denote the ideal associated to the point  $P_i \times Q_j \in X$ . For each  $(i, j) \in D$ , let  $m_{ij}$  be a positive integer if  $P_i \times Q_j \in X$ , otherwise, let  $m_{ij} = 0$ . Then we denote by  $Z$  the subscheme of  $\mathbb{P}^1 \times \mathbb{P}^1$  defined by the saturated bihomogeneous ideal

$$I_Z = \bigcap_{(i,j) \in D} I_{P_i \times Q_j}^{m_{ij}}$$

where  $I_{P_i \times Q_j}^0 := (1)$ . We say  $Z$  is a **fat point scheme** or a **set of fat points** of  $\mathbb{P}^1 \times \mathbb{P}^1$ . The integer  $m_{ij}$  is called the **multiplicity** of the point  $P_i \times Q_j$ . We shall sometimes denote the scheme as  $Z = \{(P_i \times Q_j; m_{ij}) \mid (i, j) \in D\}$ , or as  $Z = m_{11}(P_1 \times Q_1) + \dots + m_{ab}(P_a \times Q_b)$ . The **support** of  $Z$ , written  $\text{Supp}(Z)$ , is the set of points  $X$ .

A fat point scheme is said to be **arithmetically Cohen-Macaulay** (ACM for short) if the associated coordinate ring is Cohen-Macaulay. We will need a classification of ACM fat point schemes of  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  due to the first author (see [5]).

We begin by recalling a construction of [5]. Let  $Z$  be a fat point scheme in  $\mathbb{P}^1 \times \mathbb{P}^1$  where  $Z = \{(P_i \times Q_j; m_{ij}) \mid 1 \leq i \leq a, 1 \leq j \leq b\}$  with  $m_{ij} \geq 0$ . For each  $h \in \mathbb{N}$ , and for each tuple  $(i, j)$  with  $1 \leq i \leq a$  and  $1 \leq j \leq b$ , define

$$t_{ij}(h) := (m_{ij} - h)_+ = \max\{0, m_{ij} - h\}.$$

The set  $\mathcal{S}_Z$  is then defined to be the set of  $b$ -tuples

$$\mathcal{S}_Z = \{(t_{11}(h), \dots, t_{1b}(h)), (t_{21}(h), \dots, t_{2b}(h)), \dots, (t_{a1}(h), \dots, t_{ab}(h)) \mid h \in \mathbb{N}\}.$$

The elements of  $\mathcal{S}_Z$  belong to  $\mathbb{N}^b$ . Let  $\succeq$  denote the partial order where  $(i_1, \dots, i_b) \succeq (j_1, \dots, j_b)$  if and only if  $i_\ell \geq j_\ell$  for all  $\ell = 1, \dots, b$ . Then we have [5, Theorem 4.8]:

**Theorem 2.1.** *Let  $Z$  be a fat point scheme in  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then  $Z$  is ACM if and only if the elements of  $\mathcal{S}_Z$  can be totally ordered by  $\succeq$ , i.e.,  $\mathcal{S}_Z$  has no incomparable elements.*

**Remark 2.2.** Recall that the bigraded **Hilbert function** of  $Z$  is defined by  $H_Z(i, j) = \dim_k R_{i,j} - \dim_k (I_Z)_{i,j}$ . When  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  is an ACM fat point scheme, then  $H_Z(i, j)$  can be computed for all  $(i, j)$  directly from the set  $\mathcal{S}_Z$  (see [5] for details).

**Example 2.3.** We illustrate these ideas with the following set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ :

$$Z =$$

	$Q_1$	$Q_2$	$Q_3$	$Q_4$
$P_1$	5	4	2	2
$P_2$	5	3	2	1
$P_3$	4	3	1	
$P_4$	2	1		
$P_5$	1			

Consider the multiplicities 5, 4, 2, and 2 on the first ruling, i.e., the points whose first coordinate is  $P_1$ . From the construction of  $\mathcal{S}_Z$ , the tuples  $(5, 4, 2, 2), (4, 3, 1, 1), (3, 2, 0, 0), (2, 1, 0, 0), (1, 0, 0, 0)$  all belong to  $\mathcal{S}_Z$ . Notice we successively subtract one from each entry, until we reach a zero. We carry out this procedure for each ruling to find:

$$\begin{aligned} \mathcal{S}_Z = & \{(5, 4, 2, 2), (4, 3, 1, 1), (3, 2, 0, 0), (2, 1, 0, 0), (1, 0, 0, 0), (5, 3, 2, 1), \\ & (4, 2, 1, 0), (3, 1, 0, 0), (2, 0, 0, 0), (1, 0, 0, 0), (4, 3, 1, 0), (3, 2, 0, 0), \\ & (2, 1, 0, 0), (1, 0, 0, 0), (2, 1, 0, 0), (1, 0, 0, 0), (1, 0, 0, 0)\}. \end{aligned}$$

The partial order  $\succeq$ , when restricted to the set  $\mathcal{S}_Z$ , is a total ordering, i.e., there are no incomparable elements. Thus, the set  $Z$  is an ACM set of fat points.

**Remark 2.4.** We note that the following two schemes of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ :

$Z =$

	$Q_1$	$Q_2$	$Q_3$	$Q_4$
$P_1$	5	4	2	2
$P_2$	5	3	2	1
$P_3$	4	3	1	
$P_4$	2	1		
$P_5$	1			

$Z' =$

	$Q'_1$	$Q'_2$	$Q'_3$	$Q'_4$
$P'_1$	1	2	0	0
$P'_2$	3	4	1	0
$P'_3$	4	5	2	2
$P'_4$	3	5	2	1
$P'_5$	0	1	0	0

are both ACM, and reordering in a suitable way the lines of type  $(1, 0)$  and  $(0, 1)$  they become the same. Thus, if  $Z$  is ACM, we can always suppose that  $(m_{i1}, \dots, m_{ib}) \succeq (m_{j1}, \dots, m_{jb})$  for  $1 \leq i < j \leq a$ .

When  $Z$  is ACM, there are some relative bounds on the multiplicities:

**Lemma 2.5.** *Let  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  be an ACM set of fat points.*

- (i) *Suppose that there exists  $i, k, j, l$  such that  $P_i \times Q_j, P_i \times Q_l, P_k \times Q_j$ , and  $P_k \times Q_l$  all belong to  $\text{Supp}(Z)$  and let  $m_{ij}, m_{il}, m_{kj}$ , and  $m_{kl}$  be the corresponding nonzero multiplicities. Then  $m_{ij} \leq m_{il} + m_{kj} - m_{kl} + 1$ .*
- (ii) *Suppose that there exists  $i, k, j, l$  such that  $P_i \times Q_j, P_i \times Q_l$ , and  $P_k \times Q_j$ , all belong to  $\text{Supp}(Z)$ , but  $P_k \times Q_l \notin \text{Supp}(Z)$  and let  $m_{ij}, m_{il}$ , and  $m_{kj}$  be the corresponding nonzero multiplicities. Then  $m_{il} \leq m_{ij} - m_{kj} + 1$ .*

*Proof.* (i) Since  $Z$  is ACM, using Remark 2.4 we can always reorder the lines of type  $L_{P_i}$  and  $L_{Q_j}$  so that  $m_{ij} \geq m_{il}$  and  $m_{ij} \geq m_{kj}$ . Suppose that  $m_{ij} > m_{il} + m_{kj} - m_{kl} + 1$ . From

the construction of  $\mathcal{S}_Z$ , the tuples

$$(\star, m_{ij} - m_{il} + m_{kl} - 1, \star, m_{il} - m_{il} + m_{kl} - 1, \star) \text{ and } (\star, m_{kj}, \star, m_{kl}, \star)$$

are elements of  $\mathcal{S}_Z$  where  $\star$  denotes other elements of the tuple. But since  $m_{ij} - m_{il} + m_{kl} - 1 > m_{kj}$  but  $0 \leq m_{il} - m_{il} + m_{kl} - 1 = m_{kl} - 1 < m_{kl}$ , these tuples of  $\mathcal{S}_Z$  will be incomparable, which contradicts the ACM property of  $Z$ .

(ii) The proof is similar to (i). Suppose that  $m_{il} > m_{ij} - m_{kl} + 1$ . In  $\mathcal{S}_Z$ , we will have tuples of the form  $(\star, m_{ij} - (m_{ij} - m_{kl} + 1), \star, m_{il} - (m_{ij} - m_{kl} + 1), \star)$  and  $(\star, m_{kj}, \star, 0, \star)$ . But then  $\mathcal{S}_Z$  will have incomparable elements since  $m_{ij} - (m_{ij} - m_{kl} + 1) < m_{kj}$  but  $m_{il} - (m_{ij} - m_{kl} + 1) > 0$ .  $\square$

In the sequel we will require a bigraded version of Bezout's theorem:

**Theorem 2.6.** *Let  $F, G \in k[x_0, x_1, y_1, y_0]$  be two bihomogeneous forms such that  $G$  is irreducible,  $\deg G = (a, b)$ , and  $\deg F = (c, d)$ . If the curves defined by  $F$  and  $G$  in  $\mathbb{P}^1 \times \mathbb{P}^1$  meet at more than  $ad + bc$  points (counting multiplicities), then  $F = GF'$ .*

**2.2. Separators of fat points.** In [10], the authors introduced the notion of a separator for a set of fat points in  $\mathbb{P}^n \times \mathbb{P}^m$ . We recall these results, but specialize to the case of  $\mathbb{P}^1 \times \mathbb{P}^1$ . Note, we will denote a point of  $\mathbb{P}^1 \times \mathbb{P}^1$  simply by  $P$  instead  $P_i \times Q_j$ .

**Definition 2.7.** Let  $Z = m_1 P_1 + \cdots + m_i P_i + \cdots + m_s P_s$  be a set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$ . We say that  $F$  is a **separator of the point  $P_i$  of multiplicity  $m_i$**  if  $F \in I_{P_i}^{m_i-1} \setminus I_{P_i}^{m_i}$  and  $F \in I_{P_j}^{m_j}$  for all  $j \neq i$ .

If we let  $Z' = m_1 P_1 + \cdots + (m_i - 1) P_i + \cdots + m_s P_s$ , then a separator of the point  $P_i$  of multiplicity  $m_i$  is also an element of  $F \in I_{Z'} \setminus I_Z$ . The set of minimal separators are defined in terms of the ideals  $I_{Z'}$  and  $I_Z$ .

**Definition 2.8.** A set  $\{F_1, \dots, F_p\}$  is a set of **minimal separators of  $P_i$  of multiplicity  $m_i$**  if  $I_{Z'}/I_Z = (\overline{F_1}, \dots, \overline{F_p})$ , and there does not exist a set  $\{G_1, \dots, G_q\}$  with  $q < p$  such that  $I_{Z'}/I_Z = (\overline{G_1}, \dots, \overline{G_q})$ .

Important for this paper is the following definition:

**Definition 2.9.** The **degree of the minimal separators of  $P_i$  of multiplicity  $m_i$** , denoted  $\deg_Z(P_i)$ , is the tuple

$$\deg_Z(P_i) = (\deg F_1, \dots, \deg F_p) \text{ where } \deg F_i \in \mathbb{N}^2$$

and  $F_1, \dots, F_p$  is any set of minimal separators of  $P_i$  of multiplicity  $m_i$ .

For a general fat point scheme  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ , there is no known formula for  $p = |\deg_Z(P)|$ . However, as a special case of Theorem 4.3 and Theorem 5.1 of [10], we can compute the exact value for  $p$  if we also assume that  $Z$  is ACM (as we shall assume throughout the next section):

**Theorem 2.10.** *Let  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  be an ACM set of fat points. If  $P$  is a fat point of multiplicity  $m$  of  $Z$ , then  $|\deg_Z(P)| = m$ .*

## 3. MAIN RESULTS

**3.1. Fat points on a ruling.** We begin by looking at a special case, namely,  $\text{Supp}(Z) = \{P \times Q_1, \dots, P \times Q_b\}$ , i.e., all the points have the same first coordinate. The fact that these schemes are ACM follows directly from Theorem 2.1. We first require a lemma which depends upon the Hilbert functions of these schemes (see [7, Theorem 2.2]).

**Lemma 3.1.** *Let  $Z$  be a set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  of the form*

$$Z = m_1(P \times Q_1) + m_2(P \times Q_2) + \dots + m_b(P \times Q_b).$$

*Let  $m = \max\{m_j\}_{j=1}^b$ . For  $\ell = 0, \dots, m-1$ , set  $c_\ell = \sum_{p=1}^b (m_p - \ell)_+$ . If  $(i, j) \not\prec (\ell, c_\ell)$  for all  $\ell \in \{0, \dots, m-1\}$ , then  $\dim_k(I_Z)_{i,j} = 0$ .*

**Theorem 3.2.** *Let  $Z$  be a set of fat points in  $\mathbb{P}^1 \times \mathbb{P}^1$  of the form*

$$Z = m_1(P \times Q_1) + m_2(P \times Q_2) + \dots + m_i(P \times Q_i) + \dots + m_b(P \times Q_b),$$

*i.e., each point of  $\text{Supp}(Z)$  has the same first coordinate. Fix an  $i \in \{1, \dots, b\}$ , and set*

$$b_\ell = \sum_{p=1}^b (m_p - \ell)_+ \text{ for } \ell = 0, \dots, m_i - 1.$$

*Then  $\deg_Z(P \times Q_i) = \{(\ell, b_\ell - 1) \mid \ell = 0, \dots, m_i - 1\}$ .*

*Proof.* By Theorem 2.10, we have  $|\deg_Z(P \times Q_i)| = m_i$ . We first construct  $m_i$  separators  $F_0, \dots, F_{m_i-1}$  of  $P \times Q_i$  of multiplicity  $m_i$  where  $\deg(F_\ell) = (\ell, b_\ell - 1)$  for  $\ell = 0, \dots, m_i - 1$ . Our second step is to prove that these separators form a set of minimal separators.

To simplify our notation, let  $R = L_P$  and  $L_i = L_{Q_i}$  for  $i = 1, \dots, b$ . Fix an  $\ell \in \{0, \dots, m_i - 1\}$ , and let

$$A_\ell = R^\ell \text{ and } B_\ell = L_1^{(m_1-\ell)_+} L_2^{(m_2-\ell)_+} \dots L_i^{(m_i-\ell)_+-1} \dots L_b^{(m_b-\ell)_+}$$

We then set  $F_\ell = A_\ell B_\ell$ . By construction,  $\deg(F_\ell) = (\ell, b_\ell - 1)$ .

We now show that  $F_\ell \in I_{Z'} \setminus I_Z$  where  $Z'$  denotes the set of fat points with the multiplicity of  $P \times Q_i$  reduced by one. Note that  $F_\ell = R^\ell L_i^{(m_i-\ell)_+-1} F'_\ell$ , and since  $\ell + (m_i - \ell)_+ - 1 = m_i - 1$ , we have  $F_\ell \notin I_Z$  since  $F_\ell \in I_{P \times Q_i}^{m_i-1} \setminus I_{P \times Q_i}^{m_i}$ . This follows from the fact that  $F'_\ell$  does not pass through  $P \times Q_i$ . Now let  $P \times Q_f$  be any other point in  $\text{Supp}(Z)$  distinct from  $P \times Q_i$ . Since  $I_{P \times Q_f} = (R, L_f)^{m_f}$ , and since the exponents of  $R$  and  $L_f$  in  $F_\ell$  sum up to at least  $m_f$ , we have that  $F_\ell \in I_{P \times Q_f}^{m_f}$ . So  $F_\ell \in I_{Z'} \setminus I_Z$ .

Let us now show that the  $F_\ell$  are minimal separators. Let  $G$  be any separator of  $P \times Q_i$  of multiplicity  $m_i$  with  $\deg(G) = (c, d)$ . We want to show that  $\deg(G) \succeq (\ell, b_\ell - 1)$  for some  $\ell \in \{0, \dots, m_i - 1\}$ . If we can verify this fact, then the explicit separators described above would form a minimal set of minimal separators.

Suppose that for every  $\ell$ ,  $(c, d) \not\prec (\ell, b_\ell - 1)$ . Since  $Z'$  is also a fat point scheme of points on a line, by Lemma 3.1  $\dim_k(I_{Z'})_{c,d} = 0$  ( $b_\ell - 1$  appears as some  $c_\ell$  because we have reduced the multiplicity of  $m_i$  by 1). So,  $0 \neq G \in (I_{Z'})_{c,d} = (0)$ , a contradiction.  $\square$

**Remark 3.3.** By swapping the roles of the grading, we can prove a similar result for sets of points whose second coordinate are the same. We leave it to the reader to write out the corresponding statement of Theorem 3.2.

**3.2. Separators of ACM fat points.** The main result of this paper is a formula to compute the degree of a minimal separator for each fat point in an ACM fat point scheme in  $\mathbb{P}^1 \times \mathbb{P}^1$ .

**Theorem 3.4.** *Let  $Z \subseteq \mathbb{P}^1 \times \mathbb{P}^1$  be an ACM set of fat points. For any  $P_i \times Q_j \in \text{Supp}(Z)$ , let*

$$Y = m_{1j}(P_1 \times Q_j) + \cdots + m_{ij}(P_i \times Q_j) + \cdots + m_{aj}(P_a \times Q_j)\}$$

*be all the fat points of  $Z$  whose support has  $P_i$  as its first coordinate, and let*

$$W = m_{i1}(P_i \times Q_1) + \cdots + m_{ij}(P_i \times Q_j) + \cdots + m_{ib}(P_i \times Q_b)\}$$

*be all the fat points of  $Z$  whose support has  $Q_j$  as its second coordinate. Set*

$$a_\ell = \sum_{s=1}^a (m_{sj} - \ell)_+ \text{ and } b_\ell = \sum_{p=1}^b (m_{ip} - \ell)_+ \text{ for } \ell = 0, \dots, m_{ij} - 1.$$

*Then*

$$\deg_Z(P_i \times Q_j) = \{(a_{m_{ij}-1-\ell} - 1, b_\ell - 1) \mid \ell = 0, \dots, m_{ij} - 1\}.$$

Before proving this result, let us illustrate how to use it.

**Example 3.5.** We continue to use the example of Example 2.3. For convenience, we recall that

$$Z = \begin{array}{c} \begin{array}{cccc} & Q_1 & Q_2 & Q_3 & Q_4 \\ P_1 & \bullet & \bullet & \bullet & \bullet \\ P_2 & \bullet & \bullet & \bullet & \bullet \\ P_3 & \bullet & \bullet & \bullet & \\ P_4 & \bullet & \bullet & & \\ P_5 & \bullet & & & \end{array} \end{array}$$

We will compute  $\deg_Z(P_3 \times Q_2)$ . The multiplicity of  $P_3 \times Q_2$  is  $m_{3,2} = 3$ , so we will have  $|\deg_Z(P_3 \times Q_2)| = 3$ . In the notation of Theorem 3.4, we have

$$Y = 4(P_1 \times Q_2) + 3(P_2 \times Q_2) + 3(P_3 \times Q_2) + 1(P_4 \times Q_2)$$

and

$$W = 4(P_3 \times Q_1) + 3(P_3 \times Q_2) + 1(P_3 \times Q_3).$$

We now calculate  $a_0, a_1, a_2$ , and  $b_0, b_1, b_2$ :

$$\begin{array}{ll} a_0 &= 4 + 3 + 3 + 1 & b_0 &= 4 + 3 + 1 \\ a_1 &= 3 + 2 + 2 + 0 & b_1 &= 3 + 2 + 0 \\ a_2 &= 2 + 1 + 1 + 0 & b_2 &= 2 + 1 + 0. \end{array}$$

We thus get

$$\deg_Z(P_3 \times Q_2) = \{(a_0 - 1, b_2 - 1), (a_1 - 1, b_1 - 1), (a_2 - 1, b_0 - 1)\} = \{(10, 2), (6, 4), (3, 7)\}.$$

Furthermore, as we will describe in the proof of Theorem 3.4, we can explicitly determine these minimal separators. If  $L_{P_i}$  denotes the degree  $(1, 0)$  form that passes through  $P_i$  and  $L_{Q_j}$  denotes the degree  $(0, 1)$  form that passes through  $Q_j$ , then the forms  $F_1 = L_{P_1}^4 L_{P_2}^3 L_{P_3}^2 L_{P_4} L_{Q_1}^2$ ,  $F_2 = L_{P_1}^3 L_{P_2}^2 L_{P_3} L_{Q_1}^3 L_{Q_2}$  and  $F_3 = L_{P_1}^2 L_{P_2} L_{Q_1}^4 L_{Q_2}^2 L_{Q_3}$  are the minimal separators of  $P_3 \times Q_2$  of multiplicity  $m_{3,2} = 3$  with the required degrees.

*Proof.* (of Theorem 3.4) Fix a point  $P_i \times Q_j$  of multiplicity  $m_{ij}$  in  $Z$ . There are two main steps. First, we construct  $m_{ij}$  separators  $F_0, \dots, F_{m_{ij}-1}$  of  $P_i \times Q_j$  of multiplicity  $m_{ij}$  where  $\deg(F_\ell) = (a_{m_{ij}-\ell-1} - 1, b_\ell - 1)$  for  $\ell = 0, \dots, m_{ij} - 1$ . Second, we prove that these separators form a set of minimal separators of  $P_i \times Q_j$  of multiplicity  $m_{ij}$ .

To simplify our notation slightly, let  $L_i$  denote the degree  $(1, 0)$  for that passes through  $P_i$  for  $i = 1, \dots, a$ , and let  $R_j$  denote the degree  $(0, 1)$  form that passes through  $Q_j$  for  $j = 1, \dots, b$ . Fix an  $\ell \in \{0, \dots, m_{ij} - 1\}$ , and let

$$A_\ell = L_1^{(m_{1j} - (m_{ij} - \ell - 1))_+} L_2^{(m_{2j} - (m_{ij} - \ell - 1))_+} \dots L_i^{(m_{ij} - (m_{ij} - \ell - 1))_+ - 1} \dots L_a^{(m_{aj} - (m_{ij} - \ell - 1))_+}$$

and

$$B_\ell = R_1^{(m_{i1} - \ell)_+} R_2^{(m_{i2} - \ell)_+} \dots R_j^{(m_{ij} - \ell)_+ - 1} \dots R_b^{(m_{ib} - \ell)_+}$$

We then set  $F = F_\ell = A_\ell B_\ell$ . By construction,  $\deg(F) = (a_{m_{ij}-\ell-1} - 1, b_\ell - 1)$ .

We now need to show that  $F \in I_{Z'} \setminus I_Z$  where  $Z'$  denotes the set of fat points with the multiplicity of  $P_i \times Q_j$  reduced by one. Note that  $F = L_i^{(m_{ij} - (m_{ij} - \ell - 1))_+ - 1} R_j^{(m_{ij} - \ell)_+ - 1} F'$ , and since

$$(m_{ij} - (m_{ij} - \ell - 1))_+ - 1 + (m_{ij} - \ell)_+ - 1 = m_{ij} - 1,$$

we have  $F \notin I_Z$  since  $F \in I_{P_i \times Q_j}^{m_{ij}-1} \setminus I_{P_i \times Q_j}^{m_{ij}}$ . This is because  $F'$  does not pass through  $P_i \times Q_j$ .

Now take any other point  $P_e \times Q_f$  in the support of  $Z$  distinct from  $P_i \times Q_j$ . We need to show that  $F \in I_{P_e \times Q_f}^{m_{ef}}$ . Since  $I_{P_e \times Q_f} = (L_e, R_f)^{m_{ef}} = (L_e^u R_f^v \mid u + v = m_{ef})$ , it will suffice to show that the exponents of  $L_e$  and  $R_f$  in  $F$  sum up to at least  $m_{ef}$ . We break this problem into a number of cases. Recall that since we are assuming that  $Z$  is ACM, we can assume that  $m_{ab} \geq m_{cb}$  if  $a < c$  and  $m_{ab} \geq m_{ad}$  if  $b < d$ .

**Case 1:**  $e < i$  and  $f < j$ .

We have  $F = L_e^{(m_{ej} - (m_{ij} - \ell - 1))_+} R_f^{(m_{if} - \ell)_+} F'$ . We have

$$(m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{if} - \ell)_+ = m_{ej} + m_{if} - m_{ij} + 1$$

because  $m_{ej} \geq m_{ij}$  and  $m_{if} - \ell \geq m_{ij} - \ell \geq 1$ . But by Lemma 2.5, we have  $m_{ej} + m_{if} - m_{ij} + 1 \geq m_{ef}$ .

**Case 2:**  $e < i$  and  $f = j$ .

We observe that  $F = L_e^{(m_{ej} - (m_{ij} - \ell - 1))_+} R_j^{(m_{ij} - \ell)_+ - 1} F'$ . So, we have

$$(m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{ij} - \ell)_+ - 1 = m_{ej} - m_{ij} + \ell + 1 + m_{ij} - \ell - 1 = m_{ej}$$

since  $m_{ej} \geq m_{ij}$  and  $m_{ij} - \ell \geq 1$ .

**Case 3:**  $e < i$  and  $j < f \leq b$ .

In this case,  $F = L_e^{(m_{ej} - (m_{ij} - \ell - 1))_+} R_f^{(m_{if} - \ell)_+} F'$ . There are two possibilities: (a)  $m_{if} - \ell \geq 0$ , and (b)  $m_{if} - \ell < 0$ . If (a) holds, then since  $m_{ej} \geq m_{ij}$

$$(m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{if} - \ell)_+ = m_{ej} + m_{if} - m_{ij} + 1 \geq m_{ef}$$

where the last inequality holds by Lemma 2.5. If (b) holds, then  $R_f$  does not appear as a factor of  $F$ . But  $(m_{ej} - (m_{ij} - \ell - 1))_+ = m_{ej} - m_{ij} + \ell + 1 > m_{ej} - m_{ij} + m_{if} + 1 \geq m_{ef}$ , where the last inequality again holds by Lemma 2.5.

**Case 4:**  $e < i$  and  $f > b$ .

In this case,  $R_f$  is not a factor of  $F$ , and so  $F = L_e^{(m_{ej} - (m_{ij} - \ell - 1))_+} F'$ . Note that since  $f > b$ , the points  $P_{ej}$ ,  $P_{ef}$ , and  $P_{ij}$  are in the support of  $Z$ , but  $P_{if}$  is not. By Lemma 2.5, we have  $m_{ej} - m_{ij} + 1 \geq m_{ef}$ . So  $(m_{ej} - (m_{ij} - \ell - 1))_+ \geq m_{ef}$ .

**Case 5:**  $e = i$ .

If  $e = i$ , then  $f \in \{1, \dots, \hat{j}, \dots, b\}$ , and furthermore,  $F = L_i^{(m_{ij} - (m_{ij} - \ell - 1))_+ - 1} R_f^{(m_{if} - \ell)_+} F'$ . If  $m_{if} - \ell \geq 0$ , then  $(m_{ij} - (m_{ij} - \ell - 1))_+ - 1 + (m_{if} - \ell)_+ = m_{if}$ . If  $m_{if} - \ell < 0$ , then  $(m_{ij} - (m_{ij} - \ell - 1))_+ - 1 + (m_{if} - \ell)_+ = \ell > m_{if}$ .

**Case 6:**  $i < e \leq a$  and  $f < j$ .

This case is similar to Case 3.

**Case 7:**  $i < e \leq a$  and  $f = j$ .

In this situation,  $F = L_e^{(m_{ej} - (m_{ij} - \ell - 1))_+} R_j^{(m_{ij} - \ell)_+ - 1} F'$ . There are two possibilities. If  $(m_{ej} - (m_{ij} - \ell - 1)) < 0$ , then the sum of the exponents is simply  $m_{ij} - \ell - 1 > m_{ej}$ . On the other hand, if  $(m_{ej} - (m_{ij} - \ell - 1)) \geq 0$ , the sum of the exponents is  $(m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{ij} - \ell)_+ - 1 = m_{ej}$ , as desired.

**Case 8:**  $i < e \leq a$  and  $f > j$ .

We have  $F = L_e^{(m_{ej} - (m_{ij} - \ell - 1))_+} R_f^{(m_{if} - \ell)_+} F'$ . We need to consider four possibilities:

- (a)  $(m_{ej} - (m_{ij} - \ell - 1)) \geq 0$  and  $(m_{if} - \ell) \geq 0$ . In this case, the exponents sum to  $m_{ej} + m_{if} - m_{ij} + 1 \geq m_{ef}$  by Lemma 2.5;
- (b)  $(m_{ej} - (m_{ij} - \ell - 1)) \geq 0$  and  $(m_{if} - \ell) < 0$ . In this case, the exponents sum to  $m_{ej} - m_{ij} + \ell + 1 > m_{ej} + m_{if} - m_{ij} + 1 \geq m_{ef}$  by Lemma 2.5;
- (c)  $(m_{ej} - (m_{ij} - \ell - 1)) < 0$  and  $(m_{if} - \ell) \geq 0$ . We know that  $m_{ef} \leq m_{ej} + m_{if} - m_{ij} + 1$ . But  $m_{ej} - m_{ij} < -\ell - 1$ . So  $m_{ef} < m_{if} - \ell + 1 - 1 = m_{if} - \ell$ , as desired.
- (d)  $(m_{ej} - (m_{ij} - \ell - 1)) < 0$  and  $(m_{if} - \ell) < 0$ . This case cannot occur. If it did, then we would have  $0 > (m_{ej} - (m_{ij} - \ell - 1))_+ + (m_{if} - \ell)_+ = m_{ej} + m_{if} - m_{ij} + 1 \geq m_{ef} \geq 0$ .

**Case 9:**  $a > e$  and  $f < j$ .

This case is the same as Case 4.

These nine cases now show that each  $F_\ell$  with  $\ell \in \{0, \dots, m_i - 1\}$  is a separator of  $P_i \times Q_j$  of multiplicity  $m_{ij}$ . We now demonstrate that these are the minimal separators.

Let  $F$  be any separator of  $P_i \times Q_j$  of multiplicity  $m_{ij}$  with  $\deg(F) = (c, d)$ . To simplify our notation, set  $m = m_{ij}$ . We want to show that  $\deg(F) \succeq (a_{m-1-\ell} - 1, b_\ell - 1)$  for some  $\ell \in \{0, \dots, m - 1\}$ . If we can verify this fact, then the explicit separators described above would form a set of minimal separators of  $P_i \times Q_j$  of multiplicity  $m$ .



So, suppose that for every  $\ell$ ,  $(c, d) \not\geq (a_{m-1-\ell} - 1, b_\ell - 1)$ . Note, however, that  $F$  is a separator of  $P_i \times Q_j$  of multiplicity of  $m$  of  $Y$ . So, there exists a  $t$  such that  $(c, d) \succeq (a_{m-1-t} - 1, m - 1 - t)$  by Remark 3.3 and Theorem 3.2. However, since  $(c, d) \not\geq (a_{m-1-t} - 1, b_t - 1)$ , this implies that  $d < b_t$ . On the other hand, by Theorem 3.2, there exists a  $k$  such that  $(c, d) \succeq (k, b_k - 1)$  since  $F$  is also a separator of  $P_i \times Q_j$  of multiplicity of  $m$  of  $W$ . We thus have that  $b_k - 1 \leq d < b_t - 1$ , whence  $t < k$ .

Since  $b_k < b_{k-1} < \dots < b_t$ , there exists a  $p$  such that  $b_{p+1} - 1 \leq d < b_p - 1$  with  $t \leq p < k$ . Now  $(c, d) \succeq (k, b_k - 1)$ , and  $k \geq p + 1$  and  $d \geq b_{p+1} - 1$ , so we also have  $(c, d) \succeq (p + 1, b_{p+1} - 1)$ . However, because  $(c, d) \not\geq (a_{m-1-(p+1)} - 1, b_{p+1} - 1)$ , we have  $c < a_{m-p-2} - 1$ .

Consider the line  $L_i$ , i.e., the degree  $(1, 0)$  form that contains  $W$ . Then the curves defined by  $F$  and  $L_i$  meet at  $m_{i1} + m_{i2} + \dots + (m_{ij} - 1) + \dots + m_{ib} = b_0 - 1$  points. By Bezout's Theorem (see Theorem 2.6), since  $L_i$  is irreducible, and  $c \cdot 0 + d \cdot 1 = d < b_p - 1 < b_0 - 1$ , we have that  $F = F_0 L_i$ . But now consider  $F_0$ . This is a form of degree  $(c - 1, d)$ , and the curve it defines meets  $L_i$  at

$$(m_{i1} - 1)_+ + (m_{i2} - 1)_+ + \dots + (m_{ij} - 1 - 1)_+ + \dots + (m_{ib} - 1)_+ \geq b_1 - 1$$

points (the inequality comes from the fact that  $(m_{ij} - 1 - 1)_+ \geq [(m_{ij} - 1)_+ - 1]$ ). By Bezout's Theorem  $F_0 = F_1 L_i$  because  $(c - 1) \cdot 0 + d \cdot 1 = d < b_p - 1 < b_1 - 1$ . We can continue this argument until will arrive at  $F = F_p L_i^{p+1}$ , where  $\deg(F_p) = (c - p - 1, d)$ .

The form  $F_p$  and the degree  $(0, 1)$  form  $R_j$  which contains  $Y$  meet at

$$m_{1j} + m_{2j} + \dots + (m_{ij} - p - 2)_+ + \dots + m_{aj} \geq a_0 - p - 2$$

points, counting multiplicities. To see this, note that  $L_i^{p+1}$  already passes through the point  $P_i \times Q_j$   $(p + 1)$  times. Since  $F$  passes through  $P_i \times Q_j$  with multiplicity  $m - 1$ ,  $F_p$  must pass through  $P_i \times Q_j$   $(m - p - 2)_+$  times. The inequality comes from the fact that  $(m_{ij} - p - 2)_+ \geq m_{ij} - p - 2$ .

Now since  $c < a_{m-p-2} - 1$ , we have  $c - p - 1 < a_{m-p-2} - p - 2 < \dots < a_0 - p - 2$ . So, by Bezout's Theorem (Theorem 2.6), we get  $F_p = G_0 R_j$ . But then  $G_0$  has degree  $(c - p - 1, d - 1)$ , and we can repeat the above argument to show that  $G_0 = G_1 R_j$ . Continuing in this fashion, we arrive at  $F_p = G_{m-p-2} R_j^{m-p-1}$ .

We therefore have  $F = L_i^{p+1} R_j^{m-p-1} G_{m-p-2}$  where  $G_{m-p-2}$  has degree  $(c - p - 1, d - m + p + 1)$ . The exponents of  $L_i$  and  $R_j$  sum to  $m$ , which means that  $F \in (L_i, R_j)^m$ , which contradicts the fact that a separator of  $P_i \times Q_j$  of multiplicity  $m = m_{ij}$  belongs to  $(L_i, R_j)^{m-1} \setminus (L_i, R_j)^m$ . Thus, there cannot be a separator  $F$  with degree  $(c, d) \not\geq (a_{m-1-\ell} - 1, b_\ell - 1)$  for all  $\ell$ .  $\square$

**Application 3.6.** We sketch out how one might use Theorem 3.4 to compute some Hilbert functions. When  $Z$  is an ACM fat point scheme in  $\mathbb{P}^1 \times \mathbb{P}^1$ , then  $H_Z(i, j)$  can be computed for all  $(i, j)$  directly from the set  $\mathcal{S}_Z$  introduced in Section 2 (see [5] for complete details). If we pick any fat point  $P_i \times Q_j$  in  $Z$  of multiplicity  $m_{ij}$ , then by Theorem 3.4, we can compute

$$\deg_Z(P_i \times Q_j) = ((c_1, d_1), \dots, (c_{m_{ij}}, d_{m_{ij}})).$$

Let  $Z'$  be the scheme formed by reducing the multiplicity of  $P_i \times Q_j$  by one. As shown in [10, Corollary 4.4], we can compute the Hilbert function of  $Z'$  as follows:

$$H_{Z'}(r, s) = H_Z(r, s) - |\{(c, d) \in \deg(P_i \times Q_j) | (c, d) \preceq (r, s)\}|.$$

In other words, if  $Z'$  is any fat point scheme (possibly not ACM) which has the property that if we increase the multiplicity of one of its points by one to get an ACM scheme, then the Hilbert function of  $Z'$  can be computed directly from numerical information describing  $Z'$ .

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